Enumerators of lambda terms are reducing constructively

Henk Barendregt

Faculty of Mathematics and Computer Science
Catholic University
Nijmegen, The Netherlands
and
Département de Mathématique et d'Informatique
Ecole Normale Supérieure
Paris, France

Dedicated in friendship to my teacher
Dirk van Dalen
on the occasion of his 60^{th} birthday

Abstract

A closed λ-term E is called an *enumerator* if

$$\forall M \in \Lambda^{\circ} \exists n \in \mathsf{N}\ \mathsf{E}\ulcorner n \urcorner =_{\beta} M.$$

Here Λ° is the set of closed λ-terms, N is the set of natural numbers and the $\ulcorner n \urcorner$ are the Church's numerals $\lambda fx.f^n x$. Such an E is called *reducing* if moreover

$$\forall M \in \Lambda^{\circ} \exists n \in \mathsf{N}\ \mathsf{E}\ulcorner n \urcorner \twoheadrightarrow_{\beta} M.$$

In 1983 I conjectured that every enumerator is reducing. An ingenious recursion theoretic proof of this conjecture by Statman is presented in Barendregt [1992]. The proof is not intuitionistically valid, however. Dirk van Dalen has encouraged me to find intuitionistic proofs whenever possible. In the lambda calculus this is usually not difficult. In this paper an intuitionistic version of Statmans proof will be given. It took me somewhat longer to find it than in other cases.

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1. Introduction

If we have proved in Heytings arithmetic **HA** that E is an enumerator, then by Statmans result we can prove in Peano's arithmetic **PA** that E is reducing. The statement that a combinator is a reducing enumerator is Π^0_2. Therefore, by a well-known result, see e.g. Troelstra and van Dalen [1988], proposition 3.3.5 (ii), it follows that also in **HA** one can prove that E is reducing. So the reader may wonder why we give an intuitionistic proof of Statmans theorem. The first reason is that there is a difference between knowing that a statement A can be proved intuitionistically and having an intuitionistic proof. By Kreisels result we have a general recipe for transforming any proof D_{PA} in **PA** of a Π^0_2-statement into a proof D_{HA} in **HA**. But in order to obtain D_{HA} in this way, we first have to write down a formalized proof of A and then apply the recipe. The result is a formal proof but may not be understandable. The second reason is that by using Kreisels general recipe one only obtains the validity of the rule

$$\vdash_{HA} \mathsf{E} \text{ is an enumerator } \Rightarrow \ \vdash_{HA} \mathsf{E} \text{ is a reducing enumerator.}$$

A concrete **HA** proof of a statement A may be such that it also shows the implication within **HA**:

$$\vdash_{HA} \mathsf{E} \text{ is an enumerator } \rightarrow \ \mathsf{E} \text{ is a reducing enumerator.}$$

Indeed our constructive proof will yield the validity of this direct implication.

Statmans result is stronger than just stated. He showed in **PA** the following. Let $\mathcal{A} \subseteq \Lambda^\circ$ be an r.e. set. Suppose

$$\forall M{\in}\Lambda^\circ \exists N{\in}\mathcal{A}\ N =_\beta M. \tag{1}$$

Then

$$\forall M{\in}\Lambda^\circ \exists N{\in}\mathcal{A}\ N \twoheadrightarrow_\beta M. \tag{2}$$

By applying this to the set $\mathcal{A} = \{\mathsf{E}\ulcorner n \urcorner \mid n{\in}\mathsf{N}\}$ one obtains his result concerning enumerators E. We will prove

$$\vdash_{HA} (1) \ \rightarrow \ (2).$$

2. Statmans proof

We use lambda calculus notation from Barendregt [1984] and recursion theoretic notations from Rogers [1967]. In particular if ψ is a partial recursive function, then $\psi(n)\downarrow$ means that $\psi(n)$ is defined and $\psi(n)\uparrow$ means that $\psi(n)$ is undefined. A set $A \subseteq \mathsf{N}$ is called recursively enumerable (r.e.) if for some partial recursive $\psi{:}\mathsf{N}{\rightarrow}\mathsf{N}$ one has $A = \mathrm{dom}(\psi)$, i.e. $\forall n{\in}\mathsf{N}\ [n{\in}A \Leftrightarrow \psi(n)\downarrow]$. In the following the reader is supposed to know some elementary properties of r.e. sets. For example, that if A and its complement are both r.e., then A is recursive; moreover, that there exists a set $K \subseteq \mathsf{N}$ that is r.e. but not recursive.

2.1. LEMMA. *Let $M\in\Lambda$. Then there is an $M_1\in\Lambda$ in β-nf such that $M_1\mathsf{I} \twoheadrightarrow_\beta M$ and* $\mathrm{FV}(M) = \mathrm{FV}(M_1)$. *Here* $\mathsf{I} \equiv \lambda x.x$.

Proof. By induction on the structure of M we define M_1 in the following table.

M	M_1
x	$\lambda z.zx$
PQ	$\lambda z.z(zP_1z)(zQ_1z)$
$\lambda x.P$	$\lambda zx.zP_1z$

Then by induction it follows that $M_1\mathsf{I} \twoheadrightarrow_\beta M$ and $\mathrm{FV}(M) = \mathrm{FV}(M_1)$. ∎

Remember that a term $M\in\Lambda$ is *of order 0* if for no $P\in\Lambda$ one has $M =_\beta \lambda x.P$. For example $(\lambda x.xx)(\lambda x.xx)$ is of order 0.

2.2. LEMMA. (i) *For every partial recursive function ψ there is a term $F\in\Lambda^\circ$ such that for all $n\in\mathsf{N}$ one has*

$$\begin{array}{lcl}\psi(n)\downarrow & \Rightarrow & F\ulcorner n\urcorner =_\beta \ulcorner\psi(n)\urcorner\\ \psi(n)\uparrow & \Rightarrow & F\ulcorner n\urcorner \text{ is of order 0.}\end{array}$$

(ii) *Let $K \subseteq \mathsf{N}$ be an r.e. set. Then for some $P_K\in\Lambda^\circ$ one has for all $n\in\mathsf{N}$*

$$\begin{array}{lcl}n\in K & \Rightarrow & P_K\ulcorner n\urcorner \twoheadrightarrow_\beta \mathsf{I};\\ n\notin K & \Rightarrow & P_K\ulcorner n\urcorner \text{ is of order 0.}\end{array}$$

Proof. (i) Inspection of the usual proof of the λ-definability of the partial recursive functions shows that in case the function is undefined on an argument the representing λ-term is of order 0 on the corresponding numeral. For another proof due to Statman, see Barendregt [1992a].

(ii) Let $K = \mathrm{dom}(\psi)$. Let ψ be λ-defined by F. Then take $P_K \equiv \lambda c.Fc\mathsf{I}\mathsf{I}$, noting that for Church's numerals one has $\ulcorner n\urcorner\mathsf{I}\mathsf{I} =_\beta \mathsf{I}$. ∎

2.3. THEOREM (Statman [1987]). *Let $\mathcal{A} \subseteq \Lambda^\circ$ (after coding) be an r.e. set. Suppose*

$$\forall M\in\Lambda^\circ\exists N\in\mathcal{A}\ N =_\beta M. \tag{3}$$

Then

$$\forall M\in\Lambda^\circ\exists N\in\mathcal{A}\ N \twoheadrightarrow_\beta M. \tag{4}$$

Proof. Assume (3). Suppose towards a contradiction that (4) does not hold, i.e. for some $M_0\in\Lambda^\circ$

$$\forall N\in\mathcal{A}\ N \not\twoheadrightarrow_\beta M_0. \tag{5}$$

Using lemma 2 construct a term M_1 in β-nf such that $M_1\mathsf{I} \twoheadrightarrow_\beta M_0$. Let $P = P_K$ as in lemma 2 for some non-recursive r.e. set K. Define a predicate R on N as follows:

$$R(n) \iff \exists N{\in}\mathcal{A}\exists Q{\in}\Lambda\, [P\ulcorner n\urcorner \twoheadrightarrow_\beta Q \;\&\; N \twoheadrightarrow_\beta QM_1\mathsf{I}].$$

Note that R is an r.e. predicate. Claim

$$R(n) \iff n{\notin}K.$$

As to ($\Rightarrow$), suppose $R(n)$, i.e. for some $N{\in}\mathcal{A}$ and $Q{\in}\Lambda$ one has

$$P\ulcorner n\urcorner \twoheadrightarrow_\beta Q \text{ and } N \twoheadrightarrow_\beta QM_1\mathsf{I}.$$

If $n{\in}K$, then $\mathsf{I} =_\beta P\ulcorner n\urcorner =_\beta Q$, so by the Church-Rosser theorem $Q \twoheadrightarrow_\beta \mathsf{I}$ and therefore $N \twoheadrightarrow_\beta \mathsf{I}M_1\mathsf{I} \twoheadrightarrow_\beta M_0$, contradicting (5). Therefore $n{\notin}K$ and we are done. As to ($\Leftarrow$), suppose $n{\notin}K$. Then $P\ulcorner n\urcorner$ is of order 0. By (3) there is an $N{\in}\mathcal{A}$ such that $N =_\beta P\ulcorner n\urcorner M_1\mathsf{I}$. By the Church-Rosser theorem there is a common reduct L of N and $P\ulcorner n\urcorner M_1\mathsf{I}$. Since $P\ulcorner n\urcorner$ is of order 0 and M_1, I are in nf one must have $L \equiv QM_1\mathsf{I}$ with $P\ulcorner n\urcorner \twoheadrightarrow_\beta Q$. Therefore $R(n)$.

From the claim it follows that the complement of K is r.e., hence recursive (since K is itself r.e.) contradicting the choice of K. ■

What is happening here? Given $\mathcal{A}$ and a term M, we want to construct a term $N{\in}\mathcal{A}$ such that $N \twoheadrightarrow M$. We know that there is a term $N_n = P_nM_1\mathsf{I}$, with $P_n \equiv P_K\ulcorner n\urcorner$. Now

$$\begin{array}{rcl} n{\in}K & \Rightarrow & P_n \twoheadrightarrow \mathsf{I}; \\ n{\notin}K & \Rightarrow & N_n \text{ is of order } 0 \\ & \Rightarrow & N_n \twoheadrightarrow P'_nM_1\mathsf{I}, \end{array}$$

for some $P'_n \twoheadleftarrow P_n$. If—in some 'dialectic' way—one would have $n{\in}K$ & $n{\notin}K$ we would be done. Indeed, then

$$N_n \twoheadrightarrow P'_nM_1\mathsf{I} \twoheadrightarrow \mathsf{I}M_1\mathsf{I} \to M_1\mathsf{I} \twoheadrightarrow M.$$

This is impossible of course. But for some e and $P'_e{\twoheadleftarrow}P_e$ one has

$$e{\in}K \;\&\; N_e \twoheadrightarrow P'_eM_1\mathsf{I},$$

because otherwise $\mathsf{N}{-}K = \{n \mid \exists P'_n{\twoheadleftarrow}P_n\ N_n \twoheadrightarrow P'_nM_1\mathsf{I}\}$; since the latter set is r.e., the negation theorem implies that K is recursive, contrary to the choice of K. Therefore one has for this e

$$N_e \twoheadrightarrow P'_eM_1\mathsf{I} \twoheadrightarrow \mathsf{I}M_1\mathsf{I} \twoheadrightarrow M.$$

3. The intuitionistic proof

The difficulty making this reasoning constructive is the following. The e to be constructed is found via the unsolvability of the halting problem. So let $K = \{n \mid \phi_n(n)\downarrow\}$ and R be an r.e. set such that $\mathsf{N}-K \subseteq R$. We want to construct an e such that $e \in R \cap K$. Now let $R = W_e = \{n \mid \phi_e(n)\downarrow\}$. Then

$$e \notin R \ \Rightarrow\ e \notin W_e \ \Rightarrow\ e \in \mathsf{N}-K \ \Rightarrow\ e \in R.$$

Therefore by reductio ad absurdum $e \in R = W_e$ and hence also $e \in K$. Intuitionistically one has only $\neg\neg(e \in R \cap K)$. By analysing why $\mathsf{N}-K \subseteq R$ we can nevertheless prove that $e \in R$ and hence $e \in R \cap K$.

3.1. Lemma. *The following is provable in* **HA**. *Let K be an r.e. set. Then for some $P = P_K \in \Lambda^\circ$ one has for all $n \in \mathsf{N}$*

$$\begin{aligned} n \in K \quad &\Rightarrow \quad P\ulcorner n \urcorner \twoheadrightarrow_\beta \mathsf{I}; \\ P\ulcorner n \urcorner \twoheadrightarrow \lambda x.M \quad &\Rightarrow \quad n \in K. \end{aligned}$$

In particular, $n \notin K \ \Rightarrow\ P\ulcorner n \urcorner$ is of order 0.

Proof. Let E be a reducing self-interpreter, e.g. the one constructed by P. de Bruin, see Barendregt [1992]. Using lemma 2 let E_1 be a β-nf such that $\mathsf{E}_1\mathsf{I} \twoheadrightarrow \mathsf{E}$. Let t be a recursive predicate such that

$$n \in K \iff \exists k\, t(n,k).$$

Let t be λ-defined by $T \in \Lambda^\circ$. By the second fixed-point theorem, see Barendregt [1984], there exists a term $H \in \Lambda^\circ$ such that

$$Hxy \twoheadrightarrow Txy(\mathsf{K}^4\mathsf{I})\langle \mathsf{I} \rangle \mathsf{E}_1 \ulcorner H \urcorner x(\mathsf{S}^+\mathsf{y}),$$

where $\langle M \rangle = \lambda x.xM$ and S^+ λ-defines the successor function. We set $P \equiv \lambda x.Hx\ulcorner 0 \urcorner$. In order to show that P satisfies the requirements, define

$$\begin{aligned} A_k^n \ &\equiv\ \mathsf{I} && \text{if } \exists k'{<}k\ t(n,k'); \\ &\equiv\ H\ulcorner n \urcorner \ulcorner k \urcorner && \text{else.} \end{aligned}$$

Claim $A_k^n \twoheadrightarrow A_{k+1}^n$. If $A_k^n \equiv \mathsf{I}$ because $\exists k'{<}k\ t(n,k')$, then also $A_{k+1}^n \equiv \mathsf{I}$ and we are done. Otherwise $A_k^n \equiv H\ulcorner n \urcorner \ulcorner k \urcorner$ because $\neg\exists k'{<}k\ t(n,k')$. Then we have the following.

Case 1. $t(n,k)$ holds. Then $T\ulcorner n \urcorner \ulcorner k \urcorner \twoheadrightarrow \mathsf{true}$ and

$$\begin{aligned} H\ulcorner n \urcorner \ulcorner k \urcorner \quad &\twoheadrightarrow \quad T\ulcorner n \urcorner \ulcorner k \urcorner(\mathsf{K}^4\mathsf{I})\langle \mathsf{I} \rangle \mathsf{E}_1 \ulcorner H \urcorner \ulcorner n \urcorner(\mathsf{S}^+\ulcorner \mathsf{k} \urcorner) \\ &\twoheadrightarrow \quad \mathsf{true}(\mathsf{K}^4\mathsf{I})\langle \mathsf{I} \rangle \mathsf{E}_1 \ulcorner H \urcorner \ulcorner n \urcorner(\ulcorner k+1 \urcorner) \\ &\twoheadrightarrow_{gk} \quad \mathsf{K}^4\mathsf{I}\mathsf{E}_1 \ulcorner H \urcorner \ulcorner n \urcorner \ulcorner k+1 \urcorner \\ &\twoheadrightarrow \quad \mathsf{I} \equiv A_{k+1}^n. \end{aligned}$$

Case 2. $t(n,k)$ does not hold. Then $T\ulcorner n \urcorner \ulcorner k \urcorner \twoheadrightarrow \mathsf{false}$ and

$$\begin{array}{rcl} H\ulcorner n\urcorner\ulcorner k\urcorner & \twoheadrightarrow & T\ulcorner n\urcorner\ulcorner k\urcorner(\mathsf{K}^4\mathsf{I})\langle\mathsf{I}\rangle\mathsf{E}_1\ulcorner H\urcorner\ulcorner n\urcorner(\mathsf{S}^+\ulcorner\mathsf{k}\urcorner) \\ & \twoheadrightarrow & \mathsf{false}(\mathsf{K}^4\mathsf{I})\langle\mathsf{I}\rangle\mathsf{E}_1\ulcorner H\urcorner\ulcorner n\urcorner(\ulcorner k+1\urcorner) \\ & \twoheadrightarrow_{gk} & \langle\mathsf{I}\rangle\mathsf{E}_1\ulcorner H\urcorner\ulcorner n\urcorner\ulcorner k+1\urcorner \\ & \twoheadrightarrow & \mathsf{E}_1\mathsf{I}\ulcorner H\urcorner\ulcorner n\urcorner\ulcorner k+1\urcorner \\ & \twoheadrightarrow & \mathsf{E}\ulcorner H\urcorner\ulcorner n\urcorner\ulcorner k+1\urcorner \\ & \twoheadrightarrow & H\ulcorner n\urcorner\ulcorner k+1\urcorner \equiv A^n_{k+1}. \end{array}$$

In the above $\twoheadrightarrow_{gk}$ means that the reduction involves at least one gk-step of completely developing all present redexes in a term. Therefore we have that

$$\sigma : P\ulcorner n\urcorner \to A^n_0 \twoheadrightarrow A^n_1 \twoheadrightarrow \ldots \twoheadrightarrow A^n_k \twoheadrightarrow \ldots$$

is a quasi-Gross-Knuth reduction path, hence by Barendregt [1984] thm.13.2.11, a cofinal reduction sequence starting with $P\ulcorner n\urcorner$. The reasoning can be carried out in **HA**.

Now suppose that $n{\in}K$. Then $t(n,k)$ for some k. Therefore

$$P\ulcorner n\urcorner \twoheadrightarrow A^n_{k+1} \equiv \mathsf{I}.$$

Suppose on the other hand that $P\ulcorner n\urcorner \twoheadrightarrow \lambda x.M$. Then by the cofinality of σ it follows that $\lambda x.M \twoheadrightarrow A^n_k$ for some k. But then $A^n_k \equiv \mathsf{I}$ is the only possibility; therefore $n{\in}K$. ∎

Now we can give the proof of the main theorem.

3.2. Theorem (Constructive version of 2). *The following is provable in* **HA**. *Let* $\mathcal{A} \subseteq \Lambda^\circ$ *be an r.e. set. Suppose*

$$\forall M{\in}\Lambda^\circ \exists N{\in}\mathcal{A}\; N =_\beta M. \tag{6}$$

Then

$$\forall M{\in}\Lambda^\circ \exists N{\in}\mathcal{A}\; N \twoheadrightarrow_\beta M. \tag{7}$$

Proof. Suppose we have (6). Given $M{\in}\Lambda^\circ$ we want to construct an $N{\in}\mathcal{A}$ such that $N \twoheadrightarrow M$. Let $K = \{n{\in}\mathsf{N} \mid \phi_n(n){\downarrow}\}$ and $P = P_K$ as in lemma 3. Define

$$R = \{n \mid \exists Q{\in}\Lambda^\circ \exists N{\in}\mathcal{A}\; N \twoheadrightarrow QM_1\mathsf{I} \;\&\; P\ulcorner n\urcorner \twoheadrightarrow Q\}.$$

Clearly R is an r.e. set. Let $R = W_e$ in the notation of Rogers [1967]. By the assumption there exists an $N{\in}\mathcal{A}$ such that $N =_\beta P\ulcorner e\urcorner M_1\mathsf{I}$. Therefore by the Church-Rosser theorem for some $L{\in}\Lambda^\circ$ one has

$$N \twoheadrightarrow L \twoheadleftarrow P\ulcorner e\urcorner M_1\mathsf{I}.$$

Case 1. In the given reduction $P\ulcorner e\urcorner M_1\mathsf{I} \twoheadrightarrow L$ the head $P\ulcorner e\urcorner$ is never reduced to a term of the form $\lambda x.T$. Then $L \equiv QM_1\mathsf{I}$ for some $Q \twoheadleftarrow P\ulcorner e\urcorner$. Then $e{\in}R = W_e$, so $e{\in}K$, hence $P\ulcorner e\urcorner = \mathsf{I}$ and therefore $Q \twoheadrightarrow \mathsf{I}$. But then

$$N \twoheadrightarrow L \equiv QM_1\mathsf{I} \twoheadrightarrow M_1\mathsf{I} \twoheadrightarrow M.$$

Case 2. In the given reduction $P\ulcorner e\urcorner M_1\mathsf{I} \twoheadrightarrow L$ the head $P\ulcorner e\urcorner$ is reduced to a term of the form $\lambda x.T$. Then by lemma 3 it follows that $e{\in}K$ so $e{\in}W_e = R$ and therefore $N' \twoheadrightarrow Q'M_1\mathsf{I}$ for some $N'{\in}\mathcal{A}$ and $Q' \twoheadleftarrow P\ulcorner e\urcorner$. Since $e{\in}K$ again we have $Q' \twoheadrightarrow \mathsf{I}$ and hence $N' \twoheadrightarrow M$. ■

References

Barendregt, H.P.

[1984] *The Lambda Calculus, its Syntax and Semantics*, revised edition, Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam.

[1992] Theoretical pearls: Enumerators of lambda terms are reducing, *J. Funct. Programming,* vol. 2(2), 233–236.

[1992a] Theoretical pearls: Representing 'undefined' in lambda calculus, *J. Funct. Programming,* vol. 2(3), 367–374.

Troelstra, A.S. and D. van Dalen

[1988] *Constructivism in Mathematics*, vol. I, North-Holland, Amsterdam.

Rogers, H.

[1967] *Theory of recursive functions and effective computability*, McGraw-Hill, New York.

Statman, R.

[1987] Two recursion theoretic problems in lambda calculus. Manuscript, Department of mathematics, Carnegie Mellon University, Pittsburgh, PA.